INTEGRAL TRANSFORMS RELATED TO A CLASS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

BY

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0. *Introduction*

We consider the differential equation:

$$\frac{d^2 y}{dx^2} - \{\lambda^2 + q(x)\}y = 0, \tag{0.1}$$

where $q(x)$ is a continuous function for all real values of x. Furthermore we assume that there exist constants a and b such that

$$q(x) - a \in \mathfrak{L}(0, \infty), \qquad q(x) - b \in \mathfrak{L}(-\infty, 0). \tag{0.2}$$

In section 1 we will construct solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of (0.1), which are characterized by the following asymptotic behaviour in $x = +\infty$ and $x = -\infty$:

$$\begin{cases} y_1(x, \lambda) \exp \sqrt{\lambda^2 + a}\, x \to 1, \\ \dfrac{d}{dx} \{y_1(x, \lambda) \exp \sqrt{\lambda^2 + a}\, x\} \to 0 \end{cases} \tag{0.3}$$

as $x \to +\infty$ and $\operatorname{Re} \sqrt{\lambda^2 + a} \geqq 0$;

$$\begin{cases} y_2(x, \lambda) \exp - \sqrt{\lambda^2 + b}\, x \to 1, \\ \dfrac{d}{dx} \{y_2(x, \lambda) \exp - \sqrt{\lambda^2 + b}\, x\} \to 0 \end{cases} \tag{0.4}$$

as $x \to -\infty$ and $\operatorname{Re} \sqrt{\lambda^2 + b} \geqq 0$.

The purpose of this paper is to prove that under certain conditions on $f(t)$ and the path of integration the following inversion formulas hold:

$$\int_{\lambda_1 - i\infty}^{\lambda_1 + i\infty} d\lambda \frac{\lambda}{W(\lambda)}\, y_2(x_0, \lambda) \int_{-\infty}^{\infty} f(t) y_1(t, \lambda) dt = \tfrac{1}{2}\pi i \{f(x_0 + 0) + f(x_0 - 0)\}, \tag{0.5}$$

and

$$\int_{\lambda_1 - i\infty}^{\lambda_1 + i\infty} d\lambda \frac{\lambda}{W(\lambda)}\, y_1(x_0, \lambda) \int_{-\infty}^{\infty} f(t) y_2(t, \lambda) dt = \tfrac{1}{2}\pi i \{f(x_0 + 0) + f(x_0 - 0)\}, \tag{0.6}$$

where $W(\lambda)$ denotes the Wronskian of $y_1(x, \lambda)$ and $y_2(x, \lambda)$, and the integrals over λ are defined as their principal values.

The exact formulation of (0.5) and (0.6) with the conditions will be given in theorem 3 in section 3. In section 2 we prove extensions for inversion formula (0.5) in the case $f(t) \equiv 0$ for $t<x_0$, and for (0.6) in the case $f(t) \equiv 0$ for $t>x_0$. These are formulated in the theorems 1 and 2. These results are used in the proofs of (0.5) and (0.6) in section 3. A corollary of these theorems given in section 2 is related to a result of Titchmarsh on the differential equation (0.1) with $q(x)$ defined and real for $0 \leqq x < \infty$ [1, chapter IX].

Applications of the inversion formulas (0.5) and (0.6) will be given in a forthcoming paper.

1. *Special solutions of equation* (0.1)

It is easily shown that the solution of the initial value problem (0.1) with (0.3) is equivalent to that of the integral equation:

$$(1.1) \quad u_1(x, \lambda) = 1 + \frac{1}{2\sqrt{\lambda^2+a}} \int_x^\infty \{1 - e^{-2\sqrt{\lambda^2+a}\,(t-x)}\}\{q(t)-a\}\, u_1(t, \lambda) dt$$

for the function

$$(1.2) \quad u_1(x, \lambda) = y_1(x, \lambda) \exp \sqrt{\lambda^2+a}\; x, \ \mathrm{Re}\sqrt{\lambda^2+a} \geqq 0,\ \lambda^2+a \neq 0,$$

occurring in (0.3). This equation has a unique solution which may be found by the method of successive approximations:

$$u_{1,0}(x, \lambda) = 1,$$

$$u_{1,n}(x, \lambda) = 1 + \frac{1}{2\sqrt{\lambda^2+a}} \int_x^\infty \{1 - e^{-2\sqrt{\lambda^2+a}\,(t-x)}\}\{q(t)-a\}\, u_{1,n-1}(t, \lambda) dt$$

$$(n = 1, 2, \ldots).$$

Putting

$$(1.3) \quad \theta_1(x) = \int_x^\infty |q(t)-a|\, dt,$$

we obtain by mathematical induction:

$$|u_{1,n}(x, \lambda) - u_{1,n-1}(x, \lambda)| \leqq \frac{1}{n!} \frac{\{\theta_1(x)\}^n}{|\sqrt{\lambda^2+a}|^n}.$$

Hence $\lim_{n\to\infty} u_{1,n}(x, \lambda) = u_1(x, \lambda)$ exists, and

$$(1.4) \quad |u_1(x, \lambda) - 1| \leqq -1 + \exp \frac{\theta_1(x)}{|\sqrt{\lambda^2+a}|},$$

and from (1.2) and (1.4):

$$(1.5) \quad |y_1(x, \lambda) - \exp - \sqrt{\lambda^2+a}\, x| \leqq |\exp - \sqrt{\lambda^2+a}\, x| \left\{ -1 + \exp \frac{\theta_1(x)}{|\sqrt{\lambda^2+a}|} \right\},$$

for $\operatorname{Re}\sqrt{\lambda^2+a} \geqq 0$, $\lambda^2+a \neq 0$. In a similar way the function

$$(1.6) \qquad u_2(x,\lambda) = y_2(x,\lambda) \exp - \sqrt{\lambda^2+b}\, x$$

is the solution of the integral equation:

$$(1.7) \qquad u_2(x,\lambda) = 1 + \frac{1}{2\sqrt{\lambda^2+b}} \int_{-\infty}^{x} \{1-e^{-2\sqrt{\lambda^2+b}(x-t)}\}\{q(t)-b\}u_2(t,\lambda)dt,$$

where $\operatorname{Re}\sqrt{\lambda^2+b} \geqq 0$, $\lambda^2+b \neq 0$. Putting

$$(1.8) \qquad \theta_2(x) = \int_{-\infty}^{x} |q(t)-b|dt,$$

we find in the same way:

$$(1.9) \qquad |u_2(x,\lambda)-1| \leqq -1 + \exp \frac{\theta_2(x)}{|\sqrt{\lambda^2+b}|},$$

$$(1.10) \qquad |y_2(x,\lambda) - \exp \sqrt{\lambda^2+b}\, x| \leqq |\exp \sqrt{\lambda^2+b}\, x| \left\{ -1 + \exp \frac{\theta_2(x)}{|\sqrt{\lambda^2+b}|} \right\},$$

for $\operatorname{Re}\sqrt{\lambda^2+b} \geqq 0$, $\lambda^2+b \neq 0$. Furthermore we need in the proofs of the formulas another special solution $y_3(x,\lambda)$ of (0.1). $y_3(x,\lambda)$ behaves like $\exp -\sqrt{\lambda^2+b}\, x$ for $x \to -\infty$, $\operatorname{Re}\sqrt{\lambda^2+b} \geqq 0$. Let c be an arbitrary real constant. Then we choose for the solution $y_3(x,\lambda)$ of (0.1) the function

$$(1.11) \qquad y_3(x,\lambda) = u_3(x,\lambda) \exp - \sqrt{\lambda^2+b}\, x,$$

where $u_3(x,\lambda)$ is the solution of the integral equation

$$(1.12) \qquad u_3(x,\lambda) = 1 + \frac{1}{2\sqrt{\lambda^2+b}} \int_{x}^{c} \{1-e^{-2\sqrt{\lambda^2+b}(t-x)}\}\{q(t)-b\}u_3(t,\lambda)\, dt.$$

Putting

$$(1.13) \qquad \theta_3(x) = \int_{x}^{c} |q(t)-b|dt,$$

we find if $x \leqq c$ that

$$(1.14) \qquad |y_3(x,\lambda) - \exp - \sqrt{\lambda^2+b}\, x| \leqq |\exp - \sqrt{\lambda^2+b}\, x| \left\{ -1 + \exp \frac{\theta_3(x)}{|\sqrt{\lambda^2+b}|} \right\}$$

for $\operatorname{Re}\sqrt{\lambda^2+b} \geqq 0$, $\lambda^2+b \neq 0$. Now we approximate the Wronskian

$$(1.15) \qquad W(\lambda) = y_1(x,\lambda)y'_2(x,\lambda) - y_2(x,\lambda)y'_1(x,\lambda)$$

as $\lambda \to \infty$ in the region D of the λ-plane where $\operatorname{Re}\sqrt{\lambda^2+a} \geqq 0$ and $\operatorname{Re}\sqrt{\lambda^2+b} \geqq 0$. From (1.2) we derive:

$$(1.16) \qquad y'_1(x,\lambda) = \{u'_1(x,\lambda) - \sqrt{\lambda^2+a}\, u_1(x,\lambda)\} \exp - \sqrt{\lambda^2+a}\, x,$$

and from (1.1):

$$u'_1(x, \lambda) = -\int_x^\infty e^{-2\sqrt{\lambda^2+a}\,(t-x)}\{q(t)-a\}\, u_1(t, \lambda)dt.$$

In view of (1.3) and (1.4) we have:

$$|u_1(t, \lambda)| \leqq \exp \frac{\theta_1(t)}{|\sqrt{\lambda^2+a}|} \leqslant \exp \frac{\theta_1(x)}{|\sqrt{\lambda^2+a}|} \text{ for } t \geqq x,$$

$$\text{(1.17)} \qquad |u'_1(x, \lambda)| \leqq \exp \frac{\theta_1(x)}{|\sqrt{\lambda^2+a}|} \int_x^\infty |q(t)-a|dt = \theta_1(x) \exp \frac{\theta_1(x)}{|\sqrt{\lambda^2+a}|}.$$

Hence $u'_1(x, \lambda) = O(1)$ as $\lambda \to \infty$ on D with x fixed.

For $y_1(x, \lambda)$ we obtain from (1.5):

$$y_1(x, \lambda) = \left\{ 1+O\left(\frac{1}{\lambda}\right)\right\} \exp - \sqrt{\lambda^2+a}\, x,$$

and from (1.16), (1.17) and (1.4):

$$y'_1(x, \lambda) = -\sqrt{\lambda^2+a} \left\{ 1+O\left(\frac{1}{\lambda}\right)\right\} \exp - \sqrt{\lambda^2+a}\, x.$$

In a similar way

$$y_2(x, \lambda) = \left\{ 1+O\left(\frac{1}{\lambda}\right)\right\} \exp \sqrt{\lambda^2+b}\, x,$$

$$y'_2(x, \lambda) = \sqrt{\lambda^2+b} \left\{ 1+O\left(\frac{1}{\lambda}\right)\right\} \exp \sqrt{\lambda^2+b}\, x.$$

Substituting in (1.15) we obtain

$$\text{(1.18)} \qquad W(\lambda) = 2\lambda \left(1+O\left(\frac{1}{\lambda}\right)\right),$$

as $\lambda \to \infty$ on D.

In the same way we obtain for the Wronskian $W_{23}(\lambda)$ of $y_2(x, \lambda)$ and $y_3(x, \lambda)$:

$$\text{(1.19)} \qquad W_{23}(\lambda) = -2\lambda \left(1+O\left(\frac{1}{\lambda}\right)\right) \text{ as } \lambda \to \infty \text{ on } \operatorname{Re} \sqrt{\lambda^2+b} \geqq 0.$$

A property of the solutions of the differential equation (0.1) which will be used later reads as follows: the function $K(x, t; \lambda)$ satisfying (0.1) as a function of x with initial conditions

$$K(t, t; \lambda) = 0, \frac{\partial}{\partial x} K(x, t; \lambda)|_{x=t} = 1,$$

is equal to

$$K(x,t;\lambda)=\frac{y^*(x,\lambda)y(t,\lambda)-y(x,\lambda)y^*(t,\lambda)}{W(y,y^*;\lambda)} \tag{1.20}$$

for any pair y and y^* of linearly independent solutions of (0.1), where $W(y, y^*; \lambda)$ is the Wronskian of y and y^*.

2. *Inversion formulas for functions vanishing in a neighborhood of* $+\infty$ *or* $-\infty$

First we prove a generalization of the inversion formula (0.5) in the case that $f(t)$ vanishes for $t<x_0$.

Theorem 1. *Let x_0, λ_0 and λ_1 be real numbers with $\lambda_1>\lambda_0$, $\lambda_1>0$. Assume $q(x)$ is continuous for $x\geqq x_0$ and there exists a constant a with*

$$q(x)-a\in\mathfrak{L}(x_0,\infty).$$

Let $y_1(x,\lambda)$ be the solution of (0.1) *satisfying* (0.3).

Let $f(t)$ be defined for $t\geqq x_0$, be of bounded variation in a right-hand neighborhood of $t=x_0$, and

$$f(t)\,e^{-\lambda_0 t}\in\mathfrak{L}(x_0,\infty). \tag{2.1}$$

H is the region in the λ-plane where $Re\sqrt{\lambda^2+a}\geqq 0$, and C is a contour in H consisting of the line $Re\,\lambda=\lambda_1$ with contingently some finite detours in H.

Let $\varphi(\lambda)$ be a function analytic in the region G to the right of C and continuous on the closure $\bar{G}$ of G with

$$\varphi(\lambda)=e^{\lambda x_0}\left(1+O\left(\frac{1}{\lambda}\right)\right) \tag{2.2}$$

as $\lambda\to\infty$ on $\bar{G}$.

Then

$$\int\limits_{\lambda_1-i\infty}^{\lambda_1+i\infty} d\lambda\,\varphi(\lambda)\int\limits_{x_0}^{\infty} y_1(t,\lambda)f(t)dt=\pi i f(x_0+0), \tag{2.3}$$

where the integral in λ, taken over C, has its principal value.

Proof: Substituting $\nu=\sqrt{\lambda^2+a}$ we see that formula (2.3) is equivalent to

$$\lim_{\mu\to\infty}\int\limits_{\sqrt{(\lambda_1-i\mu)^2+a}}^{\sqrt{(\lambda_1+i\mu)^2+a}} d\nu\int\limits_{x_0}^{\infty}\frac{\nu}{\lambda}\,\varphi(\lambda)y_1(t,\lambda)\,f(t)dt=\pi i f(x_0+0). \tag{2.4}$$

Let G_1 be the image of G under the transformation $\nu=\sqrt{\lambda^2+a}$.

Then the function

$$g(t,\nu)=\frac{\nu}{\lambda}\,\varphi(\lambda)\,y_1(t,\lambda) \tag{2.5}$$

is analytic in G_1. From (1.3) we have $\theta_1(t) \leqq \theta_1(x_0)$ for $t \geqq x_0$. Hence from (2.5), (2.2) and (1.5) we obtain:

$$g(t, \nu) = \{\exp \nu(x_0 - t)\}\{1 + \psi(t, \nu)\} \tag{2.6}$$

where

$$\psi(t, \nu) = O\left(\frac{1}{\nu}\right) \tag{2.7}$$

as $\nu \to \infty$ on G_1 uniformly in t for $t \geqq x_0$.

Let $I(t, \mu)$ be defined by

$$I(t, \mu) = \int_{\sqrt{(\lambda_1 - i\mu)^2 + a}}^{\sqrt{(\lambda_1 + i\mu)^2 + a}} g(t, \nu) d\nu. \tag{2.8}$$

From (2.6), (2.7) and Cauchy's theorem we find for $t \geqq x_0$:

$$I(t, \mu) = \left(\int_{\sqrt{(\lambda_1 - i\mu)^2 + a}}^{\infty - i\mu} - \int_{\sqrt{(\lambda_1 + i\mu)^2 + a}}^{\infty + i\mu} \right) g(t, \nu) d\nu. \tag{2.9}$$

Now we split $I(t, \mu)$ in two parts $I_1(t, \mu)$ and $I_2(t, \mu)$:

$$(2.10)\left\{\begin{aligned} I_1(t, \mu) &= \left(\int_{\sqrt{(\lambda_1 - i\mu)^2 + a}}^{\infty - i\mu} - \int_{\sqrt{(\lambda_1 + i\mu)^2 + a}}^{\infty + i\mu} \right) \exp \nu(x_0 - t) d\nu = \\ &= \frac{1}{x_0 - t} \{\exp [\sqrt{(\lambda_1 + i\mu)^2 + a}\, (x_0 - t)] - \exp [\sqrt{(\lambda_1 - i\mu)^2 + a}\, (x_0 - t)]\}, \end{aligned}\right.$$

$$I_2(t, \mu) = \left\{ \int_{\sqrt{(\lambda_1 - i\mu)^2 + a}}^{\infty - i\mu} - \int_{\sqrt{(\lambda_1 + i\mu)^2 + a}}^{\infty + i\mu} \right\} \{\exp \nu(x_0 - t)\}\, \psi(t, \nu) d\nu. \tag{2.11}$$

First we prove:

$$\lim_{\mu \to \infty} \int_{x_0}^{\infty} f(t) I_1(t, \mu) dt = \pi i f(x_0 + 0). \tag{2.12}$$

From (2.10) we have $I_1(t, \mu) = 0(e^{-\lambda_0 t})$ for $t \to \infty$ uniformly in μ. Hence by (2.1)

$$\lim_{r \to \infty} \int_{r}^{\infty} f(t) I_1(t, \mu) dt = 0 \tag{2.13}$$

uniformly in μ. Furthermore, in view of (2.10):

$$I_1(t, \mu) = 2i \frac{\sin \mu(x_0 - t)}{x_0 - t} e^{\lambda_1(x_0 - t)} + \chi(t, \mu)\, e^{\lambda_1(x_0 - t)}, \tag{2.14}$$

where

$$\chi(t, \mu) = O\left(\frac{1}{\mu}\right) \tag{2.15}$$

as $\mu \to \infty$ uniformly on $x_0 \leqq t \leqq r$. Since $f(t)$ is of bounded variation in a right-hand neighborhood of $t = x_0$ we obtain (2.12) from Dirichlet's theorem.

Next we prove:

$$\lim_{\mu \to \infty} \int_{x_0}^{\infty} f(t)\, I_2(t, \mu) dt = 0, \tag{2.16}$$

which implies with (2.12):

$$\lim_{\mu \to \infty} \int_{x_0}^{\infty} f(t)\, I(t, \mu) dt = \pi i f(x_0 + 0). \tag{2.17}$$

Again we split the integral in (2.16) into two parts $\int_{x_0}^{s}$ and $\int_{s}^{\infty}$, where s is chosen such that $f(t)$ is of bounded variation on the interval $x_0 \leqq t \leqq s$. The real and imaginary parts of $f(t)$ therefore can be written as the difference of two monotonic functions on $x_0 \leqq t \leqq s$. We will prove that

$$\int_{x_0}^{s_1} I_2(t, \mu) dt \to 0 \tag{2.18}$$

as $\mu \to \infty$ uniformly on $x_0 \leqq s_1 \leqq s$. Then applying Bonnet's mean value theorem it follows that

$$\lim_{\mu \to \infty} \int_{x_0}^{s} f(t)\, I_2(t, \mu) dt = 0. \tag{2.19}$$

To prove (2.18) we deduce from (2.11) and (2.7):

$$\left| \int_{x_0}^{s_1} I_2(t, \mu) dt \right| \leqq K \left\{ \int_{\sqrt{(\lambda_1 - i\mu)^2 + a}}^{\infty - i\mu} + \int_{\sqrt{(\lambda_1 + i\mu)^2 + a}}^{\infty + i\mu} \right\} \frac{1}{|\nu| \operatorname{Re} \nu} \{1 - \exp \operatorname{Re} \nu (x_0 - s_1)\} |d\nu|.$$

Since $0 \leqq 1 - \exp \operatorname{Re} \nu (x_0 - s_1) \leqq 1$, the last integrals tend to zero as $\mu \to \infty$ uniformly on $x_0 \leqq s_1 \leqq s$, which proves (2.18). Furthermore from (2.7) and (2.11) we have:

$$I_2(t, \mu) = e^{-\lambda_0 t}\, O\left(\frac{1}{\mu}\right)$$

as $\mu \to \infty$ uniformly on $t \geqq s$. Hence from (2.1):

$$\lim_{\mu \to \infty} \int_{s}^{\infty} f(t)\, I_2(t, \mu) dt = 0,$$

which together with (2.19) implies (2.16).

From (2.8), (2.5), (1.5) and condition (2.1) it follows that

$$\int_{x_0}^{\infty} I(t,\mu)\, f(t)dt = \int_{\sqrt{(\lambda_1-i\mu)^2+a}}^{\sqrt{(\lambda_1+i\mu)^2+a}} d\nu \int_{x_0}^{\infty} \frac{\nu}{\lambda}\, \varphi(\lambda)\, y_1(t,\lambda)\, f(t)dt.$$

Passing to the limit and using (2.17) we obtain (2.4).

From theorem 1 we may deduce a generalization of the inversion formula (0.6) in the case that $f(t)$ vanishes for $t>x_0$.

Theorem 2. *Let x_0, λ_0 and λ_1 be real numbers with $\lambda_1>\lambda_0$, $\lambda_1>0$. Assume $q(x)$ is continuous for $x\leqq x_0$ and there exists a constant b with*

$$q(x)-b \in \mathfrak{L}(-\infty, x_0).$$

Let $y_2(x,\lambda)$ be the solution of (0.1) satisfying (0.4). Let $f(t)$ be defined for $t\leqq x_0$, be of bounded variation in a left-hand neighborhood of $t=x_0$, and

$$f(t)e^{\lambda_0 t} \in \mathfrak{L}(-\infty, x_0). \tag{2.20}$$

K is the region in the λ-plane where $Re\sqrt{\lambda^2+b}\geqq 0$, and C is a contour in K consisting of the line $Re\,\lambda=\lambda_1$ with contingently some finite detours in K.

Let $\varphi(\lambda)$ be a function analytic in the region G to the right of C and continuous on the closure $\bar{G}$ of G with

$$\varphi(\lambda)=e^{-\lambda x_0}\left(1+O\left(\frac{1}{\lambda}\right)\right) \tag{2.21}$$

as $\lambda\to\infty$ on $\bar{G}$. Then

$$\int_{\lambda_1-i\infty}^{\lambda_1+i\infty} d\lambda\, \varphi(\lambda) \int_{-\infty}^{x_0} y_2(t,\lambda)\, f(t)dt = \pi i f(x_0-0), \tag{2.22}$$

where the integral in λ, taken over C, has its principal value.

Proof: If $y(x)$ is a solution of (0.1) then $y^*(x)=y(-x)$ satisfies the differential equation:

$$\frac{d^2 y^*}{dx^2} - \{\lambda^2+q^*(x)\}\, y^* = 0, \text{ where } q^*(x)=q(-x). \tag{2.23}$$

Then $a^*=b$, $y_1^*(x,\lambda)=y_2(-x,\lambda)$. If we replace x_0 by $-x_0$, then it is easily seen that theorem 1 applied to the differential equation (2.23) gives formula (2.22).

Adding the results of theorems 1 and 2 we have:

Corollary 1. *Let x_0, λ_0 and λ_1 be real numbers with $\lambda_1>\lambda_0$, $\lambda_1>0$. Assume $q(x)$ is defined and continuous for all real values of x and there exist constants a and b such that (0.2) holds.*

Let $y_1(x,\lambda)$ *and* $y_2(x,\lambda)$ *be the solutions of* (0.1) *satisfying* (0.3), (0.4) *respectively.*

Let $f(t)$ *be defined for all real values of* t, *be of bounded variation in a neighborhood of* $t=x_0$, *and*

$$f(t)e^{-\lambda_0|t|}\in\mathfrak{L}(-\infty,\infty). \tag{2.24}$$

D *is the region in the* λ*-plane where* $Re\sqrt{\lambda^2+a}\geqq 0$ *and* $Re\sqrt{\lambda^2+b}\geqq 0$, *and* C *is a contour in* D *consisting of the line* $Re\,\lambda=\lambda_1$ *with contingently some finite detours in* D.

Let $\varphi_1(\lambda)$ *and* $\varphi_2(\lambda)$ *be functions analytic in the region* G *to the right of* C *and continuous on the closure* $\bar{G}$ *of* G *with*

$$\varphi_1(\lambda)=e^{\lambda x_0}\left(1+O\left(\frac{1}{\lambda}\right)\right),\quad \varphi_2(\lambda)=e^{-\lambda x_0}\left(1+O\left(\frac{1}{\lambda}\right)\right), \tag{2.25}$$

as $\lambda\to\infty$ *on* $\bar{G}$.

Then

$$\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\Phi(\lambda)d\lambda=\pi i\,\{f(x_0+0)+f(x_0-0)\}, \tag{2.26}$$

where

$$\Phi(\lambda)=\varphi_1(\lambda)\int_{x_0}^{\infty}y_1(t,\lambda)\,f(t)dt+\varphi_2(\lambda)\int_{-\infty}^{x_0}y_2(t,\lambda)\,f(t)dt. \tag{2.27}$$

In (2.26) *the integral is taken over* C *and has the principal value.*

Corollary 2. *Suppose* $q(x)$, $y_1(x,\lambda)$ *and* $y_2(x,\lambda)$ *are the functions of section* 0. *Then in theorem* 1 *we may choose* $\varphi(\lambda)=y_2(x_0,\lambda)$ *on account of* (2.2) *and* (1.10). *In theorem* 2 *we may choose* $\varphi(\lambda)=y_1(x_0,\lambda)$ *on account of* (2.21) *and* (1.5). *In the same way in corollary* 1 *we may take* $\varphi_1(\lambda)=y_2(x_0,\lambda)$, $\varphi_2(\lambda)=y_1(x_0,\lambda)$. *Then we obtain* (2.26) *with*

$$\Phi(\lambda)=y_2(x_0,\lambda)\int_{x_0}^{\infty}y_1(t,\lambda)\,f(t)dt+y_1(x_0,\lambda)\int_{-\infty}^{x_0}y_2(t,\lambda)\,f(t)dt.$$

A related result for the differential equation (0.1) with $q(x)$ defined and real for $0\leqq x<\infty$ can be found in Chapter IX, in particular 9.6, of [1].

Remark 1. It is easily seen that the preceding results can be adapted to the case that $q(x)$ also depends analytically on λ. Then the assumption (0.2) has to be replaced by the following conditions:

$$|q(x,\lambda)-a|\leqq q_1(x)\ \text{with}\ q_1(x)\in\mathfrak{L}(0,\infty),$$

$$|q(x,\lambda)-b|\leqq q_2(x)\ \text{with}\ q_2(x)\in\mathfrak{L}(-\infty,0),$$

for $\mathrm{Re}\sqrt{\lambda^2+a}\geqq 0$ and $\mathrm{Re}\sqrt{\lambda^2+b}\geqq 0$.

3. *Proof of the inversion formulas* (0.5) *and* (0.6)

The exact formulation of (0.5) is given in

Theorem 3. *The constants a and b and the functions* $y_1(x, \lambda)$, $y_2(x, \lambda)$ *and* $W(\lambda)$ *are defined in sections* 0 *and* 1.

Let ε, λ_1 *and* x_0 *be real numbers with* $\varepsilon > 0$, $\lambda_1 > 0$. *Let* $f(t)$ *be defined for real t, be of bounded variation in a neighborhood of* $t = x_0$, *and*

$$f(t)e^{-(\lambda_1-\varepsilon)t} \in \mathfrak{L}(0, \infty), \qquad f(t)e^{-(\lambda_1+\varepsilon)t} \in \mathfrak{L}(-\infty, 0). \tag{3.1}$$

D is the region in the λ*-plane where* $Re\sqrt{\lambda^2+a} \geqq 0$ *and* $Re\sqrt{\lambda^2+b} \geqq 0$, *and C is the contour in D consisting of the line* $Re\,\lambda = \lambda_1$ *with contingently some finite detours in D such that the zeros of* $W(\lambda)'$ *are to the left of C.*

Then (0.5) *holds.*

Furthermore (0.6) *holds if the condition* (3.1) *is replaced by*

$$f(t)e^{(\lambda_1+\varepsilon)t} \in \mathfrak{L}(0, \infty), \qquad f(t)e^{(\lambda_1-\varepsilon)t} \in \mathfrak{L}(-\infty, 0). \tag{3.2}$$

In (0.5) *and* (0.6) *the integrals in* λ *are taken over C and have their principal values.*

Proof: From (1.18) it follows that $W(\lambda) \neq 0$ for sufficiently large $|\lambda|$ in D. Hence there exists a contour C which satisfies the conditions of the theorem. It is sufficient to prove the theorem for the following two cases:

$$\text{I.} \quad f(t) \equiv 0 \text{ for } t < x_0,$$

$$\text{II.} \quad f(t) \equiv 0 \text{ for } t > x_0.$$

Case I is comprised in theorem 1 with $\varphi(\lambda) = 2\lambda y_2(x_0, \lambda)/W(\lambda)$. The condition (2.2) is satisfied because of (1.18) and (1.10).

Now we restrict ourselves to case II. Introducing $K(x, t; \lambda)$ with $y = y_1(x, \lambda)$, $y^* = y_2(x, \lambda)$ in (1.20) we see that (0.5) for this case is equivalent to

$$\lim_{\mu\to\infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} d\lambda \frac{\lambda}{W(\lambda)} y_1(x_0, \lambda) \int_{-\infty}^{x_0} y_2(t, \lambda)\, f(t)dt +$$

$$+ \lim_{\mu\to\infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda d\lambda \int_{-\infty}^{x_0} K(x_0, t; \lambda)\, f(t)dt = \tfrac{1}{2}\pi i f(x_0-0).$$

By theorem 2 with $\varphi(\lambda) = 2\lambda y_1(x_0, \lambda)/W(\lambda)$ the first limit in this equation is equal to $\frac{1}{2}\pi i f(x_0-0)$.

So we have to prove that the second limit vanishes, or putting $\sigma = \sqrt{\lambda^2+b}$ that

$$\lim_{\mu\to\infty} \int_{\sqrt{(\lambda_1-i\mu)^2+b}}^{\sqrt{(\lambda_1+i\mu)^2+b}} \sigma d\sigma \int_{-\infty}^{x_0} K(x_0, t; \lambda)\, f(t)dt = 0. \tag{3.3}$$

Now

$$
(3.4)\quad \begin{cases} \displaystyle\int_{\sqrt{(\lambda_1-i\mu)^2+b}}^{\sqrt{(\lambda_1+i\mu)^2+b}} \sigma K(x_0, t; \lambda) d\sigma = \int_{-i\mu}^{i\mu} + \\ \displaystyle\qquad + \int_{\sqrt{(\lambda_1-i\mu)^2+b}}^{-i\mu} + \int_{i\mu}^{\sqrt{(\lambda_1+i\mu)^2+b}} = I_1 + I_2 + I_3. \end{cases}
$$

On the line $\operatorname{Re} \sigma = 0$, $\sigma \neq 0$, the functions $y_2(x, \lambda)$ and $y_2(x, -\lambda)$ are defined, and linearly independent. This last assertion follows from (1.10):

$$
y_2(x, \lambda) = e^{\sigma x}(1 + o(1)),\ y_2(x, -\lambda) = e^{-\sigma x}(1 + o(1)),\ \text{as } x \to -\infty.
$$

Therefore these functions may be substituted in formula (1.20) for $K(x, t; \lambda)$. Then we see that $K(x, t; \lambda)$ is an even function of λ, and consequently of σ for $\operatorname{Re} \sigma = 0$. Hence I_1 in (3.4) vanishes.

For I_2 and I_3 we use:

$$
(3.5)\qquad K(x, t; \lambda) = \frac{y_3(x, \lambda) y_2(t, \lambda) - y_2(x, \lambda) y_3(t, \lambda)}{W_{23}(\lambda)}
$$

(cf. (1.20), where in the definition of y_3 we choose $c \geqq x_0$). From (1.14), (1.10) and (1.19) we deduce:

$$
\sigma K(x_0, t; \lambda) = \tfrac{1}{2} \left\{ 1 + O\left(\frac{1}{\mu}\right) \right\} \exp \sigma(x_0 - t) - \tfrac{1}{2} \left\{ 1 + O\left(\frac{1}{\mu}\right) \right\} \exp \sigma(t - x_0),
$$

as $\mu \to \infty$ uniformly in σ on the line segments joining $-i\mu$ with $\sqrt{(\lambda_1 - i\mu)^2 + b}$, and $i\mu$ with $\sqrt{(\lambda_1 + i\mu)^2 + b}$. Hence

$$
I_2 + I_3 = \frac{1}{t - x_0} [\cos \sqrt{(\mu + i\lambda_1)^2 - b}\ (t - x_0) -
$$

$$
- \cos \sqrt{(\mu - i\lambda_1)^2 - b}\ (t - x_0)] + O\left(\frac{1}{\mu}\right) e^{(\lambda_1 + \varepsilon)(x_0 - t)}.
$$

From this, the lemma of Riemann-Lebesgue and (3.1) we obtain:

$$
\lim_{\mu \to \infty} \int_{-\infty}^{x_0} dt\, f(t) \int_{\sqrt{(\lambda_1 - i\mu)^2 + b}}^{\sqrt{(\lambda_1 + i\mu)^2 + b}} \sigma K(x_0, t; \lambda) d\sigma = 0.
$$

From (3.5), (1.10), (1.14) and (3.1) the changing of the order of integration in the last formula can be justified.

So we obtain (3.3), and the proof of (0.5) is complete.

By using the same substitutions as in the proof of theorem 2, and condition (3.2) instead of (3.1), we may deduce (0.6) from (0.5).

Remark 2. In the theorems the condition that $f(t)$ is of bounded variation in some neighborhood of $t = x_0$ may be replaced by some other condition sufficient for the convergence of an ordinary Fourier series.

Summary

Inversion formulas for integral transforms with kernels defined as solutions of differential equation (0.1) satisfying initial conditions in $\pm\infty$ are derived under $\mathfrak{L}_1$-conditions.

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